Really, Really Rational Inattention: 
Or, How I Learned to Stop Worrying and 
Love Sticky Prices*

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Abstract

We develop a general equilibrium framework where sellers post nominal prices that 
may not respond to changes in the aggregate price level — but this is an outcome, and 
not an assumption. Money is used as a medium of exchange, and hence it is natural 
that prices are posted in dollars. Due to search frictions, there is a nondegenerate price 
distribution, where low price sellers earn less per unit but make it up on the volume. 
When the money supply increases, it is not necessary for all sellers to raise prices, 
as long as the real distribution is invariant. Profit maximization is consistent with 
sellers resetting prices infrequently, or being inattentive, even though we allow them to 
change for free whenever they like. The model has many testable predictions. We get 
closed-form solutions for the price density, and determine how it depends on inflation. 
Also, when sellers change they need not all set the same price — a new posting may be 
high, low, or in between, and there can be price reductions, despite inflation. Also, the 
frequency of price changes increases with inflation, and sellers who have not changed 
for a long time are more likely to reset prices, especially during high inflation. Although 
in a sense we provide microfoundations for a key feature of Keynesian economics, the 
policy implications are very different: our theory implies classical neutrality (inflation 
matters but the price level does not). Thus, although individual prices appear to be, 
and in a real sense are, sticky, shocks to money have no real effects.

1 Introduction

We develop a framework where in equilibrium sellers post prices in nominal terms, and some 
individual prices may not respond to changes in the aggregate price level (or course, some

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prices must respond if the aggregate price level changes). As in the Lagos-Wright model, money is essential as a medium of exchange, and hence it is natural that prices are posted in dollars. As in the Burdett-Judd model, even with homogeneous sellers and buyers, due to search frictions there is a non-degenerate distribution of prices: sellers can post high or low prices and earn the same profit, since the latter yield more sales. As the money supply increases, it is not necessary for all sellers to change their prices, as long as the aggregate real distribution stays the same. Hence profit maximization is consistent with many sellers changing prices infrequently. This happens even though we allow sellers to change their price whenever they like — i.e. we do not resort to Calvo pricing — and allow them to adjust prices for free — i.e. we do not need to impose menu costs.

Although individual sellers may not have to change their nominal prices frequently—and so one could say that they can rationally afford to be somewhat inattentive to their own price, to the aggregate price level, and to monetary policy — the model nonetheless makes many testable predictions. The distribution of relative prices is pinned down, and as in some other search-based models, we actually get the closed-form solution for the density. Also, as in those models, the empirical distribution may or may not look like that predicted by a simple versions of the model, and so one may have to introduce complications like heterogeneous sellers or buyers. Moreover, although the real distribution is invariant to the nominal price level, its shape does depend on the inflation rate in a precise way that is testable against the macro data. Also, although the price distribution has a nondegenerate support, when the aggregate price level increases some sellers drop out of the support and have to adjust. Thus, the frequency of price changes tends to increase with inflation.

In addition to delivering as an equilibrium outcome that individual sellers may change their prices infrequently, even when inflation is nonzero, the theory makes several other predictions we can compare with the data. For instance, in terms of more detailed predictions, when sellers change their prices in our model they do not necessarily all set them to the same level — and in particular they do not set them as in the standard (s, S) theory. After a change, a given seller may have a high, low or medium price. We can even get some sellers

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1The baseline model in Burdett and Mortensen (1998) e.g. predicts a wage distribution that does not match the wage data very well, and so empirical work typically uses generalized versions with heterogeneous firms and workers.
to lower their prices despite positive inflation. Also, sellers who have not changed their price for a long time are more likely to change in any given period since, they are more likely to fall out of the equilibrium support of profit maximizing prices, especially in high inflation environments. All of these predictions can be taken to the data. We think we learn something when they fit, and also when they do not fit, since the model is flexible, and this gives us guidance for extensions and generalizations.

Because we have a genuine monetary theory, as we said, as a natural outcome we can generate nominal price setting, and these prices tend to be sticky. This is in sharp contrast to simply assuming that prices are set in dollars, often in models that do not even have money, and then assuming that they cannot be adjusted frequently or costlessly. There is a sense in which one might say that our theory provides a microfoundation for a key feature in many (old or new) Keynesian models that was never derived endogenously. But it is critical to understand that the policy implications are very different. Our theory is consistent with classical neutrality: although many individual nominal prices can stay fixed when there is inflation, since the real price distribution is pinned down, if there were to be a one-time shock to money and hence to the nominal price level, the nominal distribution would shift immediately and there would be no real effect.

2 The Model

2.1 The Environment

Time is discrete and indexed by $t$. Each time period is divided into two consecutive sub-periods. In the first sub-period, exchange of a produced and indivisible good takes place in an anonymous market with search frictions. We will refer to this market as the Burdett-Judd market (BJ market). In the second sub-period, exchange of an endowed good takes place in a Walrasian market. We will refer to this market as the Arrow-Debreu market (AD market). It is worth stressing that there is nothing special about this particular ordering of the two markets, nor about the assumption that the good exchanged in the BJ market is indivisible.

The economy is populated by a continuum of households with measure 1. Each household
maximizes the utility function

$$\sum_{t=0}^{\infty} \beta^t u[1[q_t = 1] + x_t],$$

(1)

Here \(q_t \in \{0, 1\}\) is the household’s consumption of the good exchanged in the BJ market (we will refer to this as the BJ good), \(x_t \in \mathbb{R}\) is the household’s consumption of the good exchanged in the AD market (we will refer to this as the AD good), \(u > 0\) is the relative utility of a unit of the BJ good, and \(\beta \in (0, 1)\) is the discount factor. Each household receives an endowment of \(\bar{\pi} > 0\) units of the AD good in every period.

In the BJ market, there is a continuum (with measure 1) of firms. Each firm chooses its price \(p \geq 0\), taking as given the amount of money, \(m_{1,t}\), that the buyers bring into the market and the cumulative distribution function, \(F_t(p)\), of prices across all other firms. Each firm can produce the good at the constant cost \(c, c \in (0, u)\). Also, in the BJ market there is a continuum of households with measure \(b_t\), where \(b_t \in [0, 1]\) is determined by free entry. The utility cost of entering the BJ market is \(k \geq 0\). Following Burdett and Judd, we assume that households observe the entire price distribution, but can only purchase the good from a random sample of firms. In particular, a household can purchase the good from two firms with probability \(\alpha_2(b_t) = (\lambda(b_t))^2\) and it can purchase the good from just one firm with probability \(\alpha_1(b_t) = 2\lambda(b_t)(1 - \lambda(b_t))\). Also, the household cannot purchase the good with probability \(\alpha_0(b_t) = (1 - \lambda(b_t))^2\). Here \(\lambda : \mathbb{R}^+ \to [0, 1]\) is a twice differentiable function of \(b\) such that \(\lambda(0) = 1, \lambda(1) = 0,\) and \(\lambda'(b) < 0\). Note that, since households are anonymous in the BJ market, all exchange takes place using the medium of exchange, fiat money. Firm’s sales revenues are thus collected in money.

At the beginning of the AD market, each household receives \(\Delta_t \in \mathbb{R}\) units of money from the firms as dividend payments, and \(T_t \in \mathbb{R}\) units of money from the government as lump-sum transfers. Then, each household chooses how many units of the AD good to purchase (or sell). Note that, since households are recognizable in the AD market, exchange may take place with either money or credit. We find useful to denote with \(\phi_t\) the number of units of the AD good that can be purchased with 1 unit of money.

The government controls the supply of fiat money. In the AD market of period \(t - 1\), the quantity of fiat money is \(M_{t-1}\). At the beginning of the AD market of period \(t\), the
government injects \((\mu - 1)M_{t-1}\) units of fiat money as lump-sum transfers to the households, i.e. \(T_t = (\mu - 1)M_{t-1}\). Therefore, during the AD market of period \(t\), the quantity of fiat money in circulation is \(M_t = \mu M_{t-1}\). For the sake of simplicity, we assume that the growth rate of money, \(\mu\), is constant over time.

### 2.2 Firm Optimization

First, consider a firm that posts the price \(p \leq \min\{m^*_t, u/\phi_t\}\) in the BJ market. The firm meets \(\alpha_1(b_t)b_t\) buyers who cannot purchase the good from any other seller. These buyers purchase the good from the firm with probability 1. Moreover, the firm meets \(2\alpha_2(b_t)b_t\) buyers who have the opportunity to purchase the good from another seller. These buyers purchase the good from the firm with probability \(1 - F_t(p)\). Therefore, the firm’s expected profits, \(\Pi_t(p)\), are equal to

\[
\Pi_t(p) = (\phi_t p - c) b_t [\alpha_1(b_t) + 2\alpha_2(b_t)(1 - F_t(p))].
\]

Next, consider a firm that posts the price \(p > \min\{m^*_t, u/\phi_t\}\) in the BJ market. The firm meets \(b_t[\alpha_1(b_t) + 2\alpha_2(b_t)]\) buyers, but it does not sell the good to any of them. Therefore, the firm’s expected profits, \(\Pi_t(p)\), are equal to zero.

In equilibrium, every price on the support, \(F_t\), of the cumulative distribution function \(F_t(p)\) maximizes the profits of the firm, i.e. \(\Pi_t(p) = \Pi_t\) for all \(p \in F_t\), and \(\Pi_t(p) \leq \Pi_t\) for all \(p \notin F_t\). As established in Burdett and Judd (1983), this equilibrium condition implies that the cumulative distribution function \(F_t(p)\) is continuous over a compact support \(F_t = [\underline{p}_t, \overline{p}_t]\). In turn, this implies that \(\Pi_t(p) = \Pi_t(\overline{p}_t)\) for every \(p \in [\underline{p}_t, \overline{p}_t]\). Hence, the c.d.f. \(F_t(p)\) is equal to

\[
F_t(p) = 1 - \frac{\alpha_1(b_t)}{2\alpha_2(b_t)} \left( \frac{\phi_t \overline{p}_t - \phi_t p}{\phi_t p - c} \right).
\]

Since \(\Pi_t(p) \leq \Pi_t(\overline{p}_t)\) for all \(p \geq \overline{p}_t\), the highest price, \(\overline{p}_t\), is equal to the monopoly price \(\min\{m^*_t, u/\phi_t\}\). And since \(F_t(\underline{p}_t) = 0\), the lowest price, \(\underline{p}_t\), is equal to

\[
\underline{p}_t = \frac{\alpha_1(b_t)\phi_t \overline{p}_t + 2\alpha_2(b_t)c}{(\alpha_1(b_t) + 2\alpha_2(b_t))\phi_t}.
\]

In the next sections, we will find it useful to measure prices in units of the AD good rather
than in units of money. We will refer to such prices as real prices. In order to compute a real price, it is sufficient to notice that \( p \) units of money are equivalent to \( p \phi_t \) units of the AD good. Therefore, the cumulative distribution function, \( H_t(z) \), of real prices in the BJ market is equal to
\[
H_t(z) = F_t(\phi_t p) = 1 - \frac{\alpha_1(b_t)}{2\alpha_2(b_t)} \left( \frac{z - z_t}{z - c} \right). \tag{5}
\]
The highest real price, \( z_t \), is equal to \( \phi_t \bar{p}_t = \min\{\phi_t m^*_t, u\} \). And the lowest real price, \( z^*_t \), is equal to
\[
z^*_t = \phi_t \underline{p} = \frac{\alpha_1(b_t)z_t + 2\alpha_2(b_t)c}{(\alpha_1(b_t) + 2\alpha_2(b_t))}. \tag{6}
\]

2.3 Household Optimization

Let \( V_t(m) \) denote the lifetime utility of a household who has \( m \) units of money at the beginning of the BJ market. Also, let \( W_t(m) \) denote the lifetime utility of a household who has \( m \) units of money at the beginning of the AD market.

2.3.1 Arrow-Debreu Market

Consider a household who enters the AD market with \( m \) units of money, and has decided to participate in the BJ market. The income of this household is \( \overline{\pi} + \phi_t(m + T + \Delta) \). This household allocates its income between consumption of the AD good, \( x_1 \), and money holdings for the BJ market, \( \hat{m}_1 \). Therefore, the household’s lifetime utility, \( W_{1,t}(m) \), is equal to
\[
W_{1,t}(m) = \max_{x_1,\hat{m}_1} -k + x_1 + \beta V_{t+1}(\hat{m}_1), \quad \text{s.t.} \quad x_1 + \phi_t \hat{m}_1 = \overline{\pi} + \phi_t(m + T + \Delta), \tag{7}
\hat{m}_1 \geq 0.
\]

It is immediate to verify that the necessary condition for an interior choice of \( \hat{m}_1 \) is
\[
\beta V'_{t+1}^- (\hat{m}_1) - \phi_t \geq 0, \quad \beta V'_{t+1}^+ (\hat{m}_1) - \phi_t \leq 0, \tag{8}
\]
where \( V'_{t+1}^- \) and \( V'_{t+1}^+ \) respectively denote the left and the right derivative of the value function \( V_{t+1} \) with respect to \( m \). It is also immediate to verify that the value function \( W_{1,t} \) is linear with respect to \( m \), i.e. \( W_{1,t}(m) = \phi_t m + W_{1,t}(0) \).
Next, consider a household who enters the AD market with \( m \) units of money and who decided to skip the next BJ market. The income of the household is \( \bar{x} + \phi_t(m + T + \Delta) \). The household allocates its income between consumption of the AD good, \( x \), and money holdings for the next AD market, \( \hat{m}_0 \). Therefore, the household’s lifetime utility, \( W_{0,t}(m) \), is equal to

\[
W_{0,t}(m) = \max_{x_0,\hat{m}_0} \quad x_0 + \beta W_{t+1}(\hat{m}_0), \\
\text{s.t.} \quad x_0 + \phi_t\hat{m}_0 = \bar{x} + \phi_t(m + T + \Delta), \\
\hat{m} \geq 0. \tag{9}
\]

It is immediate to verify that the optimal choice of money holdings is \( \hat{m}_0 = 0 \). Also, it is immediate to verify that the value function \( W_{0,t} \) is linear with respect to \( m \), i.e. \( W_{0,t}(m) = \phi_t m + W_{0,t}(0) \).

Finally, consider a household who enters the AD market and has yet to decide whether to participate in the BJ market. Its lifetime utility is

\[
W_t(m) = \max\{W_{0,t}(m), W_{1,t}(m)\} \\
= \phi_t m + \max\{W_{0,t}(0), W_{1,t}(0)\}, \tag{10}
\]

where the second line makes use of the fact that \( W_{0,t}(m) \) is equal to \( \phi_t m + W_{0,t}(0) \), and that \( W_{1,t}(m) \) is equal to \( \phi_t m + W_{0,t}(m) \). Therefore, it is immediate to see that the derivative of the value function \( W_t \) with respect to \( m \) exists and is equal to

\[
W'_t(m) = \phi_t. \tag{11}
\]

Equation (11) vindicates our initial conjecture that the marginal utility of a unit of money in the AD market is equal to \( \phi_t \).

### 2.3.2 Burdett-Judd Market

Next, consider a household who enters the BJ market with \( m \) units of money, \( m \in [0, u/\phi_t] \).

With probability \( \alpha_0(b_t) \), the household does not meet a firm and he enters the AD market with \( m \) units of money. With probability \( \alpha_1(b_t) \), the household meets exactly one firm. If the firm’s price \( p \) is smaller than \( m \), the household purchases the good and transits to the AD market with \( m - p \) units of money. If the price is greater than \( m \), the household cannot afford

\[\text{The household never finds it optimal to bring more than } u/\phi_t \text{ units of money into the BJ market.}\]
the good and he enters the AD market with \( m \) units of money. Finally, with probability \( \alpha_2(b_t) \), the household meets two firms. If the lower of the prices posted by the two firms, \( p \), is smaller than \( m \), the household purchases the good and enters the AD market with \( m - p \) units of money. Overall, the household’s lifetime utility \( V_t(m) \) is equal to

\[
V_t(m) = W_t(0) + \phi_t m + \alpha_1(b_t) \int_{\underline{p}}^{\overline{p}} 1[p \leq m][(u - \phi_t p)df_t(p)] + \alpha_2(b_t) \int_{\underline{p}}^{\overline{p}} 1[p \leq m][(u - \phi_t p)d[1 - (1 - F_t(p))^2]].
\]

(12)

The right hand side of (12) uses \( \alpha_0(b_t) = 1 - \alpha_1(b_t) - \alpha_2(b_t) \) and \( W_t(m) = W_t(0) + \phi_t m \).

The derivative of the value function \( V_t \) with respect to \( m \) exists for all \( m \neq \{\underline{p}, \overline{p}\} \). For all \( m \) strictly smaller than \( \underline{p} \) and all \( m \) greater than \( \overline{p} \), the derivative of \( V_t \) is \( V_t'(m) = \phi_t \). For all \( m \) strictly greater than \( \underline{p} \) and strictly smaller than \( \overline{p} \), the derivative of \( V_t \) is

\[
V_t'(m) = \phi_t + (u - \phi_t m)F_t'(m) [\alpha_1 + 2\alpha_2 (1 - F_t(m))].
\]

(13)

3 Equilibrium

3.1 Definition

In this paper, we restrict attention to stationary equilibria. That is, we restrict attention to equilibria in which all the real variables are constant over time, i.e. \( \tau_t = z, H_t = H, b_t = b \) for \( t = 0, 1, ... \), and all the nominal variables grow at the same rate at which the supply of money grows, i.e. \( \underline{p}_{t+1} = \mu \underline{p}_t, F_{t+1}(\mu p) = F_t(p), m_{1,t+1} = m_{1,t}\mu, 1/\phi_{t+1} = \mu/\phi_t \) for all \( t = 0, 1, ... \). The assumption of stationarity, together with the characterization of the solution to the household’s and firm’s problems, leads to the following definition of equilibrium.

**Definition 3.1:** A Stationary Monetary Equilibrium is a distribution of real prices in the BJ market, \( H \); a real monetary balance that the households carry into the BJ market, \( \tau \); and a measure of households who participate in the BJ market, \( b \). The triple \((H, \tau, b)\) satisfies the following properties:

8
(1) Profit Maximization: The real price distribution, $H$, is equal to
\[
H(z) = 1 - \frac{\alpha_1(b)}{2\alpha_2(b)} \left( \frac{\overline{z} - z}{z - c} \right). \tag{14}
\]

(2) Optimal Money Holdings: The households’ real monetary balance, $\overline{z}$, is such that
\[
\mu \geq \beta, \quad \mu \leq \beta \left[ 1 + \alpha_1(b)H'(\overline{z})(u - \overline{z}) \right]. \tag{15}
\]

(3) Optimal Entry: The measure of households, $\beta$, who participate in the BJ market is such that
\[
\text{if } b = 0, \text{ then } \Psi(0, \overline{z}) \leq 0,
\]
\[
\text{if } b = 1, \text{ then } \Psi(1, \overline{z}) \geq 0,
\]
\[
\text{if } b \in (0, 1), \text{ then } \Psi(b, \overline{z}) = 0,
\tag{16}
\]
where $\Psi(b, \overline{z})$ is defined as
\[
\Psi(b, \overline{z}) = \alpha_1(b) \int_{\overline{z}}^\infty (u - z) dH(z) + \alpha_2(b) \int_{\overline{z}}^\infty (u - z) d \left[ 1 - (1 - H(z))^2 \right] = \frac{k}{\beta} - \left( \frac{u - \overline{z}}{\beta} \right) \overline{z}. \tag{17}
\]

Condition (1) guarantees that every price posted in the BJ market is a solution to the problem of the firm, given the equilibrium price distribution, the measure of buyers who participate in the BJ market, and the real balances that each one of these buyers carries into the BJ market. Condition (2) guarantees that the real balances carried by the buyers into the BJ market solve the problem of the household. To see this, it is sufficient to notice that the two inequalities in (15) are equivalent to the conditions for the optimality of the household’s money holdings, i.e. $\beta V_{t+1}^+(\overline{z}/\phi_{t+1}) \geq \phi_t$ and $\beta V_{t+1}^-(\overline{z}/\phi_{t+1}) \leq \phi_t$. Condition (3) guarantees that the measure of households who participates in the BJ market is consistent with the solution to the household’s entry problem. To see this, it is sufficient to notice that $\Delta(b)$ is equal to the difference between the lifetime utility of a household who chooses to participate in the BJ market, $W_{1,t}(0)$, and the lifetime utility of a household who chooses to skip the BJ market, $W_{0,t}(0)$.

From the equilibrium conditions (1) and (2), it is clear that there is a strategic complementarity between the households’ choice of money holdings and the firms’ choice of prices. On the one hand, the households have no incentive to hold more money than the firms’ highest price. On the other hand, the firms have no incentive to set a price that is higher
than the amount of money that is carried around by the households. Because of this strategic complementarity, our model admits a continuum of stationary equilibria indexed by the household’s money holdings. However, not all of these equilibria are equally robust to a small modification of the environment (see Jean, Rabinovich, and Wright, 2009).

In particular, consider an environment in which the number of firms is large but finite. Suppose that a firm deviates from the equilibrium \((H, \bar{\pi}, b)\) and sets its price to \(\bar{\pi} + \epsilon\), where \(\epsilon > 0\) is an arbitrarily small number. As long as \(\mu\) is strictly smaller than \(\beta [1 + \alpha_1(b)H'(\bar{\pi})(u - \bar{\pi})]\), the households are going to increase their money holdings by epsilon in order to be able to trade with the deviant firm. Therefore, as long as \(\beta [1 + \alpha_1(b)H'(\bar{\pi})(u - \bar{\pi})]\), the profits of the deviant firm are going to increase. This argument motivates the following equilibrium refinement.

**Definition 3.2:** A Refined Stationary Monetary Equilibrium is stationary monetary equilibrium \((H, \bar{\pi}, b)\) such that \(\mu\) is equal to \(\beta [1 + \alpha_1(b)H'(\bar{\pi})(u - \bar{\pi})]\).

### 3.2 Existence and Uniqueness

First, consider the equilibrium condition (2). This condition guarantees that the household’s choice of real money balances, \(\bar{\pi}\), is optimal given the real price distribution, \(H\), and the fraction of households who participate in the BJ market, \(b\). Using the equilibrium condition (1) to substitute out the real price distribution, we can express condition (2) as

\[
i \geq 0, \quad i = \frac{\alpha_1(b)^2 u - \bar{\pi}}{2 \alpha_2(b) \bar{\pi} - c},
\]

where \(i\) denotes the nominal interest rate, \((\mu - \beta)/\beta\). The first part of (18) is always satisfied because, by assumption, the growth rate of money, \(\mu\), is greater than the discount factor, \(\beta\). The second part of (18) is satisfied if the household’s real balances are equal to

\[
\bar{\pi}_{RB}(b) = \frac{2 (1 - \lambda(b))^2 u + ic}{2 (1 - \lambda(b))^2 + i}.
\]

From (19), it follows that, for \(b = 0\), the household’s optimal real balances are equal to the firm’s cost of production, \(c\). For \(b = 1\), the household’s optimal real balances are equal to
(2u+ic)/(2+i). Moreover, for all b between 0 and 1, the household’s optimal real balances are strictly increasing with respect to b. Intuitively, the higher is the measure of buyers in the BJ market, the lower the probability that two firm compete for the same buyer, the higher the distribution of prices (in the sense of first order stochastic dominance), and the higher are the household’s optimal real balances. The green line in Figure 1 is the plot of the household’s optimal real balances as a function of the measure of households who participate in the BJ market. That is, the green line is the plot of $\bar{\tau}_{RB}(b)$.

Next, consider the equilibrium condition (3). This condition guarantees that the measure of households who participate in the BJ market, $\bar{\tau}$, is consistent with the household’s optimal entry decision given the firms’ real price distribution, $H$, and the household’s real balances, $\bar{\tau}$. Using the equilibrium condition (1) to substitute out the real price distribution, we can express the household’s surplus from entering the BJ market as

$$
\Psi(b, \bar{\tau}) = \alpha_1 \int_{\bar{\tau}}^{\tau} \frac{\alpha_1 \tau u - z}{(z-c)^2} \frac{d\tau}{z} + \alpha_2 \int_{\bar{\tau}}^{\tau} \frac{\alpha_1 \tau u - z}{(z-c)^2} \frac{d\tau}{z} - \frac{k}{\beta} - i\bar{\tau}
$$

where the second line makes use of the fact that

$$
\int_{\bar{\tau}}^{\tau} \frac{\alpha_1 \tau u - z}{(z-c)^2} \frac{d\tau}{z} = u - c - \frac{\alpha_1 \tau c}{\alpha_2} \ln \left( \frac{\alpha_1 + 2\alpha_2}{\alpha_1} \right),
$$

$$
\int_{\bar{\tau}}^{\tau} \frac{\alpha_1 \tau u - z}{(z-c)^2} \frac{d\tau}{z} = u - c - \frac{\alpha_1 \tau c}{\alpha_2} + \frac{\alpha_2 \tau c}{\alpha_2} \ln \left( \frac{\alpha_1 + 2\alpha_2}{\alpha_1} \right).
$$

From (20), it follows that the household’s surplus from entering the BJ market is strictly positive for all $\bar{\tau} < \bar{\tau}_{FE}(b)$; it is strictly negative for all $\bar{\tau} > \bar{\tau}_{FE}(b)$; and it is equal to zero for $\bar{\tau} = \bar{\tau}_{FE}(b)$, where $\bar{\tau}_{FE}(b)$ is given by

$$
\bar{\tau}_{FE}(b) = \frac{2\lambda(b)(1 - \lambda(b))u + \lambda(b)^2(u - c) - k/\beta}{i + 2\lambda(b)(1 - \lambda(b))}.
$$

From (21) and the equilibrium condition (3), it follows that the measure of households who participate in the BJ market is $b = 0$ if $\bar{\tau} \geq \bar{\tau}_{FE}(0)$, where $\bar{\tau}_{FE}(0)$ is equal to $(u - c - k/\beta)/i$. The equilibrium measure of buyers who participate in the BJ market is $b = 1$ if $\bar{\tau} \leq \bar{\tau}_{FE}(1)$, where $\bar{\tau}_{FE}(1)$ is equal to $-(k/\beta)/i$. And the equilibrium measure of buyers in the BJ market is $b \in (0, 1)$ if $\bar{\tau} = \bar{\tau}_{FE}(b)$, where $\bar{\tau}_{FE}(b)$ is strictly decreasing with respect to
Intuitively, the higher are the households’ real balances, the higher the price distribution (in the sense of first order stochastic dominance), the lower the household’s surplus from entering the BJ market, and the lower the measure of buyers who choose to enter. The red line in Figure 1 is the plot of the real balances, \( \overline{z} \), which induce \( b \) households to participate in the BJ market. That is, the red line in Figure 1 is the plot of \( \overline{z}_{FE}(b) \).

Given the properties of \( \overline{z}_{RB}(b) \) and \( \overline{z}_{FE}(b) \), it is immediate to prove that there exists one and only one \( b \) such that \( \overline{z}_{RB}(b) = \overline{z}_{FE}(b) \) for all nominal interest rates \( i \) smaller than \( \tilde{i} \), where \( \tilde{i} \) is equal to \( (u - c - k/\beta)/c \). Moreover, it is immediate to prove that there exists no \( b \) such that \( \overline{z}_{RB}(b) = \overline{z}_{FE}(b) \) for any nominal interest \( i \) strictly greater than \( \tilde{i} \). These findings lead to the following theorem.

**Proposition 3.1.** (Existence and Uniqueness) *For all \( i \in [0, \tilde{i}] \), there exists a unique refined monetary stationary equilibrium. For all \( i > \tilde{i} \), a refined monetary stationary equilibrium does not exist.*
4 The Behavior of Nominal Prices

One of the properties of the equilibrium of our model is that firms are indifferent between posting any price that falls in the interval between $\tilde{z}$ and $\bar{z}$ (when measured in real terms). For this reason, the equilibrium of our model may be able to support a pricing policy in which the firm does not change its nominal price in every period. That is, our model may be able to explain the existence (and, perhaps, the extent) of the nominal rigidities that are consistently found in the data. Interestingly, our model may be able to explain the existence of nominal rigidities even though the model does not assume the existence of either menu costs or Calvo fairies.

In this section, we want to investigate the possibility that our model is consistent with the observed behavior of prices. To this aim, we are first going to specify a parametric (and relatively flexible) family of pricing policies. The pricing policies in this family combine some of the elements of time-dependent pricing models and some of the elements of state-dependent pricing models. Second, we are going to identify which of the pricing policies in this family can be supported by the equilibrium of our model. Third, for every policy that is supported by the equilibrium, we are going to compute some of the statistics that are used in the empirical literature to measure the extent of nominal rigidities (e.g., the average duration of a price, the frequency of a price change, etc...). Finally, we are going to calibrate our model and verify whether there exists a pricing policy that is supported by the equilibrium and that can match the statistical properties of the US micro-data on prices. We are going to carry out this analysis for the case of an inflationary economy, i.e. $\mu > 1$, as this is the only empirically relevant case.

4.1 Supportable Pricing Policies

Suppose that firms adjust their prices according to the policy

$$
\begin{align*}
pt+1 &= \begin{cases} 
pt & \text{w.p. } \rho \\
\tilde{z}/\phi_{t+1} & \text{w.p. } 1 - \rho 
\end{cases} \quad \text{if } pt\phi_{t+1} \geq \tilde{z}, \\
\bar{z}/\phi_{t+1} & \quad \text{if } pt\phi_{t+1} < \tilde{z}.
\end{align*}
$$

(22)
where \( \tilde{z} \) is a random variable drawn from the cumulative distribution function \( G(z) \). According to the policy (22), a firm changes its price with certainty if the price it posted in the previous period was lower (in real terms) than the lower support of the equilibrium price distribution \( H \). Otherwise, a firm keeps its price unchanged with probability \( \rho \), and it readjusts it with probability \( 1 - \rho \). The two parameters that characterize the pricing policy (22) are the probability \( \rho \in [0, 1] \), and the cumulative distribution function \( G : [\tilde{z}, \overline{z}] \to [0, 1] \).

The pricing policy (22) is consistent with the stationary equilibrium \((H, \overline{z}, b)\) if the distribution of real prices in period \( t + 1 \) is equal to \( H \) given that all firms follow (22) and the distribution of prices in period \( t \) is \( H \). In order to find out whether the pricing policy (22) is consistent with the equilibrium, note that the fraction of real prices (and associated firms) that exits the interval \([\tilde{z}, z]\) is equal to

\[
X(z) = \begin{cases} 
H(z), & \text{if } z < \mu\tilde{z}, \\
H(\mu\tilde{z}) + (1 - \rho) [H(z) - H(\mu\tilde{z})], & \text{if } z \geq \mu\tilde{z}.
\end{cases} 
\tag{23}
\]

Conversely, the fraction of real prices (firms) that enters the interval \([\tilde{z}, z]\) is equal to

\[
I(z) = \begin{cases} 
\rho [H(\mu z) - H(\mu\tilde{z})] + M(\rho)G(z), & \text{if } z < \mu\tilde{z}, \\
\rho [H(\mu z) - H(z)] + M(\rho)G(z), & \text{if } z \geq \mu\tilde{z}.
\end{cases} 
\tag{24}
\]

where \( M(\rho) \) is defined as

\[
M(\rho) = [H(\mu z) + (1 - \rho)(1 - H(\mu\tilde{z}))]. 
\tag{25}
\]

The flows of real prices in and out of the interval \([\tilde{z}, z]\) are equal, if \( G \) is equal to

\[
G(z) = \frac{[H(z) + \rho (H(\mu\tilde{z}) - H(\mu z))] / M(\rho)}{M(\rho)}.
\tag{26}
\]

The function defined in (26) is a cumulative distribution if and only if: (a) \( G(z) = 0 \); (b) \( G(\overline{z}) = 1 \); and (c) \( G'(z) \geq 0 \) for all \( z \in [\tilde{z}, \overline{z}] \). It is immediate to verify that conditions (a) and (b) are always satisfied. After computing the derivative of \( H \) with respect to \( z \), we can rewrite \( G'(z) \) as

\[
G'(z) = \begin{cases} 
\frac{1}{M(\rho) 2 \alpha_2(b) \overline{z}^2} \left[ \frac{(z - c) - \rho \mu (z - c)}{(z - c) \mu - \overline{z} \mu} \right], & \text{if } z \leq \overline{z}/\mu, \\
\frac{1}{M(\rho) 2 \alpha_2(b) (z - c)^2}, & \text{if } z > \overline{z}/\mu.
\end{cases} 
\tag{27}
\]
From the previous expression, it is immediate to verify that $G'(z) \geq 0$ for all $z \in [\underline{z}, \overline{z}]$. Therefore, also condition (c) is satisfied. This leads to the following proposition.

**Proposition 4.1: (Supportable Pricing Policies).** The stationary equilibrium $(H, \overline{z}, b)$ supports the pricing policy (22) if and only if $\rho$ belongs to the interval $[0, 1]$ and $G(z)$ is equal to $H(z) + \rho (H(\mu_{z}) - H(\mu_{z})) / M(\rho)$.

For the pricing policies that are supported by the equilibrium, we want to compute the same statistics that are typically used in the empirical literature to measure the extent of nominal rigidities. First, we want to compute the average duration of a price. Let $N$ denote the largest integer such that $\mu^{N}_{\underline{z}} \leq \overline{z}$. In period $t$, there are $H(\mu^{n}_{\underline{z}}) - H(\mu^{n-1}_{\underline{z}})$ prices in the interval $[\mu^{n-1}_{\underline{z}}, \mu^{n}_{\underline{z}}]$ for $n = 1, 2, ... N$. In period $t$, there are also $1 - H(\mu^{N}_{\underline{z}})$ prices in the interval $[\mu^{N}_{\underline{z}}, \overline{z}]$. If a price belongs to the interval $[\mu^{n-1}_{\underline{z}}, \mu^{n}_{\underline{z}}]$ in period $t$, it will change in period $t + i$ (and not until then) with probability $\rho^{i-1}(1 - \rho)$ for $i = 1, 2, ... n - 1$, and it will change in period $t + n$ with probability $\rho^{n-1}$. If a price belongs to the interval $[\mu^{N}_{\underline{z}}, \overline{z}]$ in period $t$, it will change in period $t + i$ with probability $\rho^{i-1}(1 - \rho)$ for $i = 1, 2, ... N$, and it will change in period $t + N + 1$ with probability $\rho^{N+1}$. The average duration of a price, $A(\rho)$, is equal to

$$A(\rho) = \left\{ \sum_{n=1}^{N} [H(\mu^{n}_{\underline{z}}) - H(\mu^{n-1}_{\underline{z}})] \frac{1 - \rho^{n}}{1 - \rho} \right\} + [H(\overline{z}) - H(\mu^{N}_{\underline{z}})] \frac{1 - \rho^{N+1}}{1 - \rho} \quad (28)$$

Since the stationary equilibrium is consistent with the pricing policy (22) for any $\rho \in [0, 1]$ and $A(\rho)$ is continuous and increasing with respect to $\rho$, the stationary equilibrium is consistent with any average price duration between $A(0)$ and $A(1)$.

Next, we want to compute the frequency at which prices change. In period $t$, there are $H(\mu_{z})$ prices in the interval $[\underline{z}, \mu_{z}]$, and $1 - H(\mu_{z})$ prices in the interval $[\mu_{z}, \overline{z}]$. If a price belongs to the first interval in period $t$, it will change in period $t + 1$ with probability 1. If a price belongs to the second interval, it will change in period $t + 1$ with probability $1 - \rho$. Therefore, the frequency of a price change in period $t+1$ is equal to $H(\mu_{z}) + (1-\rho)(1-H(\mu_{z}))$, which we have denoted above as $M(\rho)$. Since the stationary equilibrium is consistent with the pricing policy (22) for any $\rho \in [0, 1]$ and $M(\rho)$ is continuous and decreasing with respect to $\rho$, the stationary equilibrium is consistent with any frequency of a price change between
Proposition 4.2. (Nominal Price Rigidity) The stationary equilibrium \((H, \bar{z}, b)\) is consistent with any average price duration in the interval \([A(0), A(1)]\). The stationary equilibrium \((H, \bar{z}, b)\) is consistent with any frequency of price adjustment in the interval \([M(1), M(0)]\).

4.2 Confronting the Data

The parameters of the model are the inflation rate, \(\mu\), the discount factor, \(\beta\), the meeting-probability function, \(\lambda(b)\), the utility cost of entering the BJ market, \(k\), the utility of the BJ good, \(u\), and the cost of producing the BJ good, \(c\). We choose the model period to be one month. We set \(\mu\) equal to 1.002 so that the annual inflation rate in equilibrium is 2.5 percent, which is the average of the US inflation rate over the period between 1991 and 2002 (the period studied by Bils and Klenow, 2005). We set \(\beta\) equal to 0.996 so that the annual real interest rate in the model is 4.8 percent, which is the average of the US interest rate on triple-A bonds over the period between 1991 and 2002. We set \(k\) equal to 0.41 so that the average mark-up in the BJ market is 15 percent, which is a number within the range of estimates of the average markup for the US economy (see, e.g., Rotemberg and Woodford 1995, and Basu and Fernald, 1997). We set \(\lambda(b) = \min\{(2b)^{-1/2}, 1\}\). We normalize the utility \(u\) to 1, and we tentatively set the cost \(c\) to 0.5.

Given these values for the parameters \(\beta, k, \lambda, u, \) and \(c\), we compute the stationary equilibrium of the model, the maximum and minimum average price durations, \(A(0)\) and \(A(1)\), and the maximum and minimum frequency of price adjustment, \(M(1)\) and \(M(0)\), for inflation rates between 0.1 and 25 percent per quarter. Figure 2 plots the minimum and the maximum average price duration as a function of the quarterly inflation rate. Figure 3 plots the minimum and the maximum frequency of price adjustment as a function of the quarterly inflation rate. As the inflation rate increases, we find that the maximum of the average price duration falls and the minimum frequency of price adjustment decreases. Intuitively, as the inflation rate increases, a given nominal price moves faster (in real terms) through the interval \([\bar{z}, \bar{z}]\) and, hence, the maximum average price duration falls and the minimum frequency of a price change increases. Since the equilibrium is always consistent with a pricing policy in which firms reset their prices in every period, the minimum average price...
duration and the maximum frequency of price adjustment do not depend on the inflation rate.

The predictions of the model are testable. Together with the minimum and the maximum frequency of price adjustment that are consistent with the model, Figure 3 presents a scatter plot of the empirical frequency of price adjustment and of the quarterly inflation rate (this scatter includes data from all of the empirical studies considered by Golosov and Lucas, 2007). From this figure, we can see that the model can account for the frequency at which US prices changed over the low-inflation period between 1991 and 2002 (see Bils and Klenow, 2005). Since the model is calibrated to the US economy, this is the only rigorous empirical test. However, we think it is encouraging that the model can also account for the frequency at which Mexican prices changed both during a high-inflation period (1995-1996) and during a low-inflation period (2000-2002) (see Gagnon, 2005). Moreover, the model can account for the frequency of price adjustment observed in the EU over the low-inflation period between 1995 and 2000 (see Dhyne et alii, 2005), and for the frequency of price adjustment observed in Israel over the period of moderate inflation between 1992 and 1993 (see Baharad and Eden, 2004). Our model only fails to account for the episodes of hyper-inflation in Israel (1981-1982) and Poland (1990-1996).
5 The Effect of Monetary Shocks

To be added.

6 Extensions

The economy described above has a number of particular characteristics which may be viewed as representing restrictive assumptions (e.g. the indivisibility of the BJ good). In fact our main qualitative findings do not hinge on several of the particular characteristics of the environment as we have presented it here. In general, any modification to the environment that preserves price dispersion and indifference of sellers across price levels at a point in time will leave these findings unaffected.

For example, it is straightforward to extend the model to allow for multiple BJ goods, and aggregate shocks to money growth and production costs can be introduced along the lines of Head, Kumar and Lapham (2008). In this section, we describe briefly two environments, both with a divisible BJ good. In the first, sellers are restricted, as in our main model, to post linear prices and in the second they are not. In both of these extensions, we fix the
number of buyers present in the BJ market. Entry could, however, be handled exactly as in our main economy presented above.

6.1 Divisibility of the BJ Good

In this extension the exchange and production in the BJ market proceed much as they do in the model of Head and Kumar (2005). That environment, however, does not have an AD market—all trade effectively takes place in the BJ market. As a result the economy’s monetary equilibrium may be significantly different from that described above. Here we show that the two environments generate similar results for the distributions of real and nominal prices in the BJ market. The AD market in this extension functions just as that in the main model described above.

6.1.1 The Environment

The is a unit measure of identical buyers, and a measure, s, of identical firms, each of which can produce any quantity of the BJ good at constant marginal cost, c. Each buyer has preferences defined over the consumption of two goods with utility given by:

$$\sum_{t=0}^{\infty} [u(q_t) + x_x],$$

(29)

where $q_t \in \mathbb{R}$ is the quantity of goods purchased by the buyer in the BJ market. Let $u$ be three times continuously differentiable, with $u(0) = 0, u' > 0,$ and $u'' < 0$. Also, assume there exists $q^* > 0$ such that $u'(q^*) = c$. Since this section is for illustrative purposes only, for brevity we restrict our analysis to constant relative risk aversion (CRRA) utility functions with relative risk aversion coefficient $\eta \equiv -\frac{u''(q)}{u'(q)} < 1$. As above, in every period, each buyer is endowed with $\bar{x} > 0$ units of the AD good.

Exchange in the BJ market follows the pattern described by Head and Kumar (2005). At the opening of the market, each firm posts nominal price $p_t$, and commits to satisfy demand at that price. Households again observe $k$ prices with probability $\alpha_k$ for $k = 1, 2$. Let $F(P)$ denote the distribution of nominal prices in the current period, and let $H(p)$ denote the distribution of real prices, where as above we will restrict attention to equilibria in which the all real quantities are constant and nominal variables grow at the rate of money creation, $\mu$. 

19
Given the price information they receive, buyers decide whether and how much to purchase. Clearly, if a buyer purchases it will do so exclusively at the lowest price it observes. Let \( J(p) \) denote the distribution of the lowest (real) price observed by a buyer:

\[
J(p) = \alpha_1 H(p) + \alpha_2 \left[ 1 - [1 - H(p)]^2 \right], \quad \forall p \in \mathcal{F}. \tag{30}
\]

### 6.1.2 Buyers’ Optimization

As above, let \( W(m) \) denote the expected value for a buyer who enters the AD market of the current period with \( m \) units of money and \( V_{+1}(\hat{m}) \) the expected value of entering the subsequent BJ market (in period \( t + 1 \)) with \( \hat{m} \) units.

1). The AD market:

A buyer chooses \( x \) and \( \hat{m} \) to solve

\[
W_i(m) = \max_{x,\hat{m}} \{ x + \beta V_{+1}(\hat{m}) \} \quad \text{s.t.} \quad x = \bar{x} + \phi (m - \hat{m}) + \Delta + T. \tag{31}
\]

The necessary condition for the optimal choice of \( \hat{m} \) is

\[
\beta V'_{+1}(\hat{m}) = \phi, \tag{32}
\]

and the envelope condition is

\[
W'(m) = \phi. \tag{33}
\]

2). The BJ market:

A buyer which enters the BJ market of the current period with \( m \) units of money solves:

\[
V(m) = W(0) + \phi m + \int_{\mathcal{F}} u \left( d(m, P) \left[ \frac{1}{P} - \phi \right] \right) dJ(P). \tag{34}
\]

where \( d(m, P) \) denotes the buyer’s optimal monetary expenditure at nominal price \( P \). It is straightforward to show that in the BJ market, given lowest price observed, \( P \), a buyer with \( m \) dollars chooses \( d(m, P) \) to solve

\[
\max_d \quad \left\{ u \left( \frac{d}{P} \right) + W_i(m - d) \right\}, \quad \forall P \quad \text{s.t.} \quad d \leq m, \quad \forall P. \tag{35}
\]
In general, the buyer’s expenditure constraint may be non-binding for some prices. Moreover, a buyer’s payoff from trade must be non-negative, i.e. \( u \left( \frac{d}{\mathcal{P}} \right) - \phi d \geq 0 \). Otherwise, the buyer would prefer to carry money into the AD market rather than spend it in the BJ market. This requirement implies reservation price, \( P^r \), such that \( u \left( \frac{m}{P^r} \right) = \phi m \). Households trade if and only if the lowest price they observe satisfies \( P \leq P^r \).

The optimal expenditure rule depends, of course, \( u \). The following proposition characterizes buyers’ optimal expenditure rule:

**Proposition 6.1:** Define the nominal price level \( \hat{P} \) such that \( u' \left( \frac{m}{\hat{P}} \right) \frac{1}{\hat{P}} = \phi \). The buyer’s optimal expenditure rule, \( d(m, P) \), has the following form:

\[
d(m, P) = \begin{cases} 
  m, & \text{if } P \leq \hat{P} \\
  \textnormal{otherwise.} & (36)
\end{cases}
\]

For a proof of this proposition see the Appendix. The cutoff price \( \hat{P} \) is that at which the buyer’s expenditure constraint binds. At this point the marginal benefit from spending one unit of money in the BJ market is at least as great as its marginal cost, i.e. its value in the subsequent AD market. Given Proposition 6.1, the derivative of \( V \) exists for all \( m \) and satisfies:

\[
V'(m) = \phi + \int_{P}^{\hat{P}} u' \left( m \left[ \frac{1}{P} - \phi \right] \right) \left[ \frac{1}{P} - \phi \right] dJ(P). \tag{37}
\]

Clearly, given the assumed properties of \( u(\cdot) \), \( V'(m) > 0 \) and \( V''(m) < 0 \).

### 6.1.3 Firm Optimization

A firm’s expected profit from posting nominal price \( P \) the BJ market is given by

\[
\pi(P) = \frac{1}{s} \left[ \left( \phi - \frac{c}{P} \right) D(M, P) \right] \left[ \alpha_1 + 2\alpha_2(1 - F(P)) \right]. \tag{38}
\]

Taking as given buyers’ optimal expenditure rule, \( D(M, P) \), as well as the common distribution of posted prices, \( F(P) \), firms choose a price, \( P \), to maximize expected profits. To ensure that the expected profits are non-negative, no firm will post a price lower than \( P^* = \frac{c}{\phi} \), which we refer to as the marginal cost price.

We also define the *monopoly* price as that which a firm would post if it knew with certainty that any buyer observing its price would not receive any other price quote (i.e.
would have no other opportunity to trade). Following the logic of Burdett and Judd (1983) it is straightforward to show that if any firm believes that all its competitors are charging prices strictly below the monopoly price, then its optimal choice is to post that price. Thus, the upper support of any distribution of prices posted by profit maximizing firms, \( \bar{P} \) must equal the monopoly price. Clearly, \( \bar{P} \leq P^r \) as posting any price higher results in an expected profit of zero. The following proposition (also proved in the Appendix) characterizes maximum price posted (i.e. the monopoly price, \( \bar{P} \)) by any profit maximizing firm:

**Proposition 6.2:** Given buyers’ optimal expenditure rule, \( D(M, P) \) (see Proposition 6.1), the optimal price posted by a monopoly is given by:

\[
\bar{P} = \max\{P^m, \bar{P}\} \quad \text{where} \quad P^m \text{ satisfies:}
\]

\[
u''[u'^{-1}(\phi P^m)] u'^{-1}(\phi P^m) + \phi P^m = c. \tag{39}
\]

and \( \bar{P} \) is as specified in Proposition 6.1.

Since no profit maximizing firm will ever post a price above the monopoly price, \( \bar{P} \), or below the marginal cost price, \( P^* \), given distribution \( F(p) \), an individual firm is indifferent between posting any price, \( P \in [p^*, \bar{P}] \), that satisfies

\[
D(M, P) \left( \phi - \frac{c}{\bar{P}} \right) [\alpha_1 + 2\alpha_2(1 - F(P))] = D(M, \bar{P}) \left( \phi - \frac{c}{\bar{P}} \right) \alpha_1. \tag{40}
\]

Using (40), it then follows that optimization by firms generates the following distribution of nominal posted prices:

\[
F(P) = 1 - \frac{\alpha_1}{2\alpha_2} \left[ \frac{D(M, \bar{P}) \left( \phi - c/\bar{P} \right)}{D(M, P) \left( \phi - c/P \right)} - 1 \right], \quad \forall P \in F = [P, \bar{P}], \tag{41}
\]

where \( \bar{P} \) is given by Proposition 6.2 and the lowest price posted, \( P \), satisfies

\[
\frac{D(M, \bar{P}) \left( \phi - c/\bar{P} \right)}{D(M, P) \left( \phi - c/P \right)} = \frac{\alpha_1 + 2\alpha_2}{\alpha_1}. \tag{42}
\]

### 6.1.4 Stationary Equilibrium

We now may define a stationary equilibrium as above, except that the number of buyers in the BJ market is now fixed, and each buyer’s expenditure rule must be determined in equilibrium:

**Definition 6.1:** A Stationary Equilibrium is a distribution of real prices in the BJ market,
a real money balance that all buyers carry into the BJ market, \( \bar{z} \); and a real expenditure rule, \( d(z, p) \) used by all buyers in the BJ market such that:

1. Taking \( H(p) \) as given, buyers carry \( \bar{z} \) and use the expenditure rule, \( d(\bar{z}, p) \) to solve their optimization problem

2. Given \( H(p) \) and \( d(\bar{z}, p) \) all prices in the support of \( H \) maximize firms’ expected profits.

Using (37) in buyers first order condition, (37) and defining the nominal interest rate,

\[
i = \frac{\phi}{\bar{\phi}_{+1}} - 1 = \frac{\gamma}{\beta} - 1,
\]

we have an expression implicitly characterizing real balances in equilibrium:

\[
i = \int_{\bar{p}}^{p} \left[ u'(\frac{d(\bar{z}, p)}{p}) \frac{1}{p} - 1 \right] dJ(p).
\]  

Thus, in a stationary equilibrium, the marginal cost of carrying additional currency into next period’s BJ market equals the discounted expected marginal utility from spending it at that time.

For brevity we omit here a proof of existence. Assuming that a solution, \( \bar{z} \), to (44) exists, however, the following can be established by straightforward calculation:

**Proposition 6.3:** Given \( \gamma > \beta \), \( \alpha_1 \in (0,1) \) and equilibrium real balances, \( \bar{z} \), the equilibrium distribution of real prices, \( \bar{H}(p) \) is given by

\[
\bar{H}(p) = 1 - \frac{\alpha_1}{2\alpha_2} \left[ \frac{D(\bar{z}, \bar{p})}{D(\bar{z}, p)} \frac{1 - c/\bar{p}}{1 - c/p} - 1 \right], \quad \forall p \in \mathcal{F},
\]

where \( p \) and \( \bar{p} \) are given by

\[
\bar{p} = \max \left[ p^m, \bar{p} \right]
\]

and

\[
\frac{d(\bar{z}, \bar{p})}{d(\bar{z}, p)} \frac{1 - c/\bar{p}}{1 - c/p} = \frac{\alpha_1 + 2\alpha_2}{\alpha_1}.
\]

In a stationary equilibrium, the real variables \( \bar{p}, p^r \) and \( p^m \) satisfy:

\[
u'(\frac{z}{\bar{p}}) \frac{1}{\bar{p}} = 1, \quad u\left(\frac{z}{p^r}\right) = z \quad \text{and} \quad u''\left[u'^{-1}(p^m)\right] u'^{-1}(p^m) + p^m = B.
\]
Similarly, in equilibrium buyers’ expenditure rule may be written:

\[ d(z, p) = \begin{cases} z, & \text{if } p \leq \hat{p} \\ p u'^{-1}(p), & \text{otherwise.} \end{cases} \quad (49) \]

## 6.2 Non-linear Pricing

### 6.2.1 The Environment

Let preferences for the BJ good be linear, \( u(q) = q \). Let the cost of producing this good (in utils) be given by \( c(q) \), where \( c' > 0, c'' > 0, c(0) = 0 \) and let there exist \( q^* \) with \( c'(q^*) = 1 \). All other aspects of the physical environment are as in other divisible goods versions.

### 6.2.2 Sellers’ optimization

Define \( \sigma = q - \phi d \) as a buyer’s surplus from exchanging \( d \) units of currency for \( q \) units of the BJ good, where \( \phi \) is the price of money in the upcoming AD market, as before. Assume that in the BJ market, firms post contracts delivering a particular surplus, \( \sigma \). Conditional on surplus offered, the firm chooses \((q, d)\) to maximize its own profit, \( \pi(\sigma) \):

\[
\pi(\sigma) = \max_{q,d} [\phi d - c(q)]
\]

subject to:

\[
q - \phi d \geq \sigma \quad (51)
\]

\[
d \leq M \quad (52)
\]

where we assume that the firm believes that all buyers carry the same amount of money. It is straightforward to show that for \( \sigma \in [0, q^*] \) the profit-maximizing contract is:

\[
(q, d) = \begin{cases} (\sigma + \phi M, M), & \sigma \leq q^* - \phi M; \\
(q^*, \frac{q^* - \sigma}{\phi}), & \sigma > q^* - \phi M. \end{cases} \quad (53)
\]

and that each executed contract results in the following realized profit to the firm:

\[
\pi(\sigma) = \begin{cases} \phi M - c(\phi M + \sigma), & \sigma \leq q^* - \phi M; \\
q^* - \sigma - c(q^*), & \sigma > q^* - \phi M. \end{cases} \quad (54)
\]

Figure 3 depicts the profit maximizing contract as a function of \( \sigma \). We can associate with each contract a nominal price, \( P(\sigma) = d/q \), and a real price, \( p(\sigma) = \phi d/q \).
In order to make a sale, a seller’s contract must be observed by a buyer who observes no other contract which offers a higher surplus. Let $F(\sigma)$ denote the CDF of offered contracts, and assume that a buyer observes one contract with probability $\alpha_1$ and two with probability $\alpha_2$. Let $b$ be the ratio of buyers to sellers. Entry would affect this, but for now just let it be fixed, say at 1. Let $G(\sigma)$ denote the distribution of the highest surplus observed:

$$G(\sigma) = \alpha_1 F(\sigma) + \alpha_2 F(\sigma)^2. \quad (55)$$

The expected number of sales by a firm that posts a contract offering surplus $\sigma$ equals:

$$s(\sigma) = b\alpha_1 + 2b\alpha_2 F(\sigma). \quad (56)$$

The value of a seller who posts such a contract is then given by:

$$V^*(\sigma) = \begin{cases} 
[\phi M - c(\phi M + \sigma)]s(\sigma), & \sigma \leq q^* - \phi M; \\
[q^* - \sigma - c(q^*)]s(\sigma), & \sigma > q^* - \phi M.
\end{cases} \quad (57)$$

For all sellers to be optimizing, each posted contract must again generate equal expected profit. With where $V^*(0) = [\phi M - c(\phi M)]b\alpha_1$, the CDF $F(\sigma)$ may be derived analytically:

$$F(\sigma) = \begin{cases} 
\frac{\alpha_1}{2\alpha_2} \left[ \frac{c(\phi M + \sigma) - c(\phi M)}{\phi M - c(\phi M) + \sigma} \right], & \sigma \leq q^* - \phi M; \\
\frac{\alpha_1}{2\alpha_2} \left[ \frac{\phi M - c(\phi M)}{q^* - \sigma - c(q^*)} - 1 \right], & \sigma > q^* - \phi M,
\end{cases} \quad (58)$$

There are thus two cases which differ with regard to the maximal surplus offered, $\bar{\sigma}$. Fixing $\phi$, $\bar{\sigma}$ exceeds $q^* - \phi M$ if

$$\frac{\alpha_1}{2\alpha_2} < \frac{\phi M - c(q^*)}{c(q^*) - c(\phi M)}. \quad (59)$$

That is, if the fraction of buyers observing two contracts is large enough. In this case $\bar{\sigma}$ is given by

$$\bar{\sigma} = \frac{[q^* - c(q^*)](\alpha_1 + \alpha_2) - [\phi M - c(\phi M)]\alpha_1}{\alpha_1 + 2\alpha_2}. \quad (60)$$

From here on we restrict attention to this case.

### 6.2.3 Buyers’ optimization

Consider now a buyer entering the BJ market with $m \leq \bar{D}$, where $\bar{D}$ is the highest expenditure required by a posted contract. Buyers have no incentive to enter with $m > \bar{D}$ (except
at the Friedman rule). This buyer’s value is

\[ V^b(m) = \phi m + \begin{cases} \int_{q* - \phi m}^\sigma \sigma dG(\sigma), & m < \bar{D}; \\ \int_0^\sigma \sigma dG(\sigma), & \sigma > \bar{D}. \end{cases} \quad (61) \]

That is, a buyer who brings in \( m \) less than \( \bar{D} \), can only accept the (high surplus) contracts that supply \( q^* \) and require \( d \leq m \). If they bring in \( \bar{D} \), then they can take any contract.

Now, it can be shown that \( \bar{m} \) exists for all \( m \neq \bar{D} \) and satisfies:

\[ V'(m) = \phi + (q^* - \phi m)G'(q^* - \phi m)\phi > 0 \quad (62) \]

and

\[ V''(m) = -\phi G'(q^* - \phi m) - (q^* - \phi m)G''(q^* - \phi m)\phi < 0 \quad (63) \]

for \( m < \bar{D} \) (with \( V' = \phi \) for \( m > \bar{D} \)).

Buyers’ first-order condition for choice of \( \hat{m} \) in the AD market of the current period is the same as above:

\[ \beta V^l_{t+1}(\hat{m}_1) - \phi \geq 0, \quad \beta V^r_{t+1}(\hat{m}_1) - \phi \leq 0, \quad (64) \]

or, for the case of \( \hat{m} < \bar{D} \)

\[ \phi \leq \beta [\phi + (q^* - \phi \hat{m})G'(q^* - \phi \hat{m})\phi] . \quad (65) \]

### 6.2.4 Stationary Equilibrium

We now associate a stationary equilibrium with a real money balance, \( \bar{z} = \phi M \), and a distribution of contracts, \( F(\sigma) \), which satisfies (58) given \( \bar{z} \). In a stationary equilibrium the first-order condition may then be written:

\[ i = (q^* - \bar{z})G'(q^* - \bar{z}), \quad (66) \]

where the nominal interest rate is again given by (43).

Let \( \Psi(\bar{z}) \equiv (q^* - \bar{z})G'(q^* - \bar{z}) - i \). Then, a stationary monetary equilibrium is associated with a solution to the equation \( \Psi(Z) = 0 \).

**Proposition 6.4:** Fix \( i \geq 0 \). Then there exists a unique stationary monetary equilibrium.
**Proof:** Note first that \( G'(\sigma) = [\alpha_1 + 2\alpha_2 F(\sigma)] F'(\sigma) \) and that for \( \sigma \in (q^* - \phi M, \bar{\sigma}) \),

\[
G'(q^* - \bar{\varepsilon}) = \frac{\alpha_1}{2\alpha_2} \left[ \frac{(\bar{\varepsilon} - c(\bar{\varepsilon}))^2}{(\bar{\varepsilon} - c(q^*))^3} \right].
\]

Thus,

- \( \Psi \) is continuous and differentiable for \( \bar{\varepsilon} \in (C(q^*), q^*) \)
- \( \Psi'(\bar{\varepsilon}) < 0 \)
- \( \lim_{\bar{\varepsilon} \to c(q^*)} \Psi(\bar{\varepsilon}) = \infty \) and \( \Psi(q^*) = -i \).

As a result, there exists a unique \( \bar{\varepsilon} \in (C(q^*), q^*) \) such that \( \Psi(\bar{\varepsilon}) = 0 \).

Suppose that given \( i \) such that \( \Psi(\bar{\varepsilon}) = 0 \), an individual buyer would like to deviate by carrying real balances \( \tilde{\varepsilon} < \bar{\varepsilon} = \phi M \). This buyer will forgo the ability to accept certain contracts, but save the marginal cost of carrying money balances, \( i \). For this to be advantageous to the buyer, it must be the case that \( V'_{+1}(\hat{m}_{+1}) < i \) for some \( \hat{m}_{+1} < M_{+1} \). Then,

\[
(q^* - \bar{\varepsilon}) \frac{\alpha_1}{2\alpha_2} \left[ \frac{(\bar{\varepsilon} - c(\bar{\varepsilon}))^2}{(\bar{\varepsilon} - c(q^*))^3} \right] < i,
\]

where \( \bar{\varepsilon} = \phi_{+1} \hat{m}_{+1} \). Since its LHS declines monotonically in \( \bar{\varepsilon} \), (68) contradicts \( \Psi(\bar{\varepsilon}) = 0 \) where \( \bar{\varepsilon} = \phi_{+1} M_{+1} \), and so there can be no such profitable deviation.

### 7 Conclusions

To be added.

### References


